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Quantum Entanglement Under Lorentz Transformation

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Abstract We study the properties of quantum entanglement in moving frames, with a non-maximally entangled bipartite state: $|\phi\rangle = \sqrt{1-\varepsilon}|\uparrow\uparrow\rangle + \sqrt{\varepsilon}|\downarrow\downarrow\rangle$, $(0 < \varepsilon < 1)$. We calculate the concurrence of this state under Lorentz transformation and show that if the momenta part of the spin-1/2 pair is appropriately correlated, the concurrence is invariant ($\mathcal{C}(\rho) = 2\sqrt{\varepsilon - \varepsilon^2}$); despite the entanglement of this state is not maximal, there is no transfer of entanglement between spin and momentum.

Keywords Quantum entanglement · Relativistic wave equations

1 Introduction

Entanglement is a novel feature of quantum physics which is absent in classical physics. Mathematically speaking entanglement is a property of multipartite quantum states that arises from the tensor product structure of the Hilbert space and the superposition principle. It demonstrates the nonlocal character of quantum mechanics which is the very basis of quantum information processing, for example quantum teleportation [1]. From the viewpoint of quantum joint measurement the truth of entanglement is nonlocal correlation collapse [2, 3].

In recent years, following the work of [4], a series of work has been attributed to studying quantum entanglement in relativistic frames, inertial or not [5–11], giving some novel properties of entanglement. For a single free spin-1/2 massive particle, the reduced density matrix for spin is not covariant under Lorentz transformations, the spin entropy is not a relativistic scalar and has no invariant meaning [5]. Consider for a fully entangled spin-1/2 Bell state, the spins' entanglement of the Bell state will be degraded When observed by a Lorentz boost observer, because Lorentz boost introduces a transfer of entanglement between degrees of freedom (spin and momentum) [6, 12].

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A more surprising fact is that for a maximally entangled Bell state of two spin-$1/2$ massive particles moving in opposite directions, the Bell's inequality holds in the ultrarelativistic limit ($\beta = v/c = 1$), though vectors lead to the maximum violation of the Bell's inequality in the nonrelativistic regime. As shown by Czachor [4], in the ultrarelativistic limit, projections of spin in the directions perpendicular to the momentum vanish for both particles and spin are (anti-) paralleled to the momentum direction. With these novel properties in moving frames Li and Du [12] suggest a relativistic invariant protocol for quantum communication by the fact that if the momenta of the spin-$1/2$ pair are appropriately entangled the entanglement between spins of the Bell'state remains maximally when viewed from any Lorentz transformed frames.

In this paper, we show that for a pair of spin-$1/2$ massive particles, if the momenta are appropriately entangled, there is no entanglement transfer from momenta to spins though the entanglement of spins is not maximal; the concurrence is invariant.

This paper is organized as follows: in Sect. 2 we briefly introduce Lorentz transformation of quantum states and Wigner representation of the Lorentz group. In Sect. 3 we calculate the concurrence of the non-maximally entangled state. Finally we give our conclusion.

2 Relativistic Transformation of Quantum States and Wigner Representation of the Lorentz Group

It is known that, finite dimensional representations of Lorentz transformations are non-unitary. The resolution to this dilemma is the representations of the Wigner rotation which is a rotation in the rest frame of the particle that leaves the rest momentum invariant. The Wigner rotation can restore unity of the transformation between relativistic single and multiparticle states [14]. Through explicit calculation of the Wigner rotation one can describe the entangled states observed from two inertial frames moving with constant relative velocity. In this paper we will follow the text [13] by Weinberg and one can find a more detailed version in [7, 12]. Note that $c = \hbar = 1$ and $\mu, \nu = \{0, 1, 2, 3\}$.

A general Poincaré transformation relates the coordinates x^μ and x'^μ of two inertial frames S and S' via

$$\begin{aligned} x'^\mu &\equiv L(\Lambda, b) = \Lambda^\mu_\nu x^\nu, \\ L(\overline{\Lambda}, \overline{b})L(\Lambda, b) &= L(\overline{\Lambda}\Lambda, \overline{\Lambda}b + \overline{b}). \end{aligned} \tag{1}$$

A Poincaré transformation $L(\Lambda, b)$ induces a liner unitary transformation $U(\Lambda)$ on the vectors in the Hilbert space of physical states [4, 9–11]

$$|\Psi\rangle \to U(\Lambda)|\Psi\rangle, \tag{2}$$

and the operator U satisfies the composition rule

$$\begin{aligned} U(\overline{\Lambda})U(\Lambda) &= U(\overline{\Lambda}\Lambda), \\ U^\dagger U &= I, \end{aligned} \tag{3}$$

where I is the unit matrix. And the single particle state operator satisfies the following transformation rule

$$U(\Lambda)|\Psi_{p,\sigma}\rangle_{AB} = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'} D^{(j)}_{\sigma'\sigma}(W(\Lambda, p))|\Phi_{p_\Lambda,\sigma}\rangle_{A'B'}. \tag{4}$$

Here, $W(\Lambda, p)$ is the Wigner's little group element given by

$$W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p), \tag{5}$$

where $D^{(j)}(W(\Lambda, p))$ is the representation of Wigner rotation for angular-j particles (for spin-$1/2$ particles $j = 1/2$). One can give the derivation of the representation of the Wigner's little group $W(\Lambda, p)$ for spin-$1/2$ particles [7] and the representation can be written as

$$D^{(1/2)}(W(\Lambda, p)) = D^{(-1/2)}(L(\Lambda p))D^{(1/2)}(\Lambda)D^{(1/2)}(L(p)). \tag{6}$$

Consider an arbitrary boost given by the velocity $\vec{v}$ with $\vec{e}$ as the normal vector in the boost direction, the Lorentz transformation Λ^{μ}_{ν} can be written as

$$\begin{aligned}
\Lambda^{i}_{j} &= \delta_{ij} + e_i e_j(\cosh\alpha - 1),\\
\Lambda^{i}_{0} &= \Lambda^{0}_{i} = e_i \sinh\alpha,\\
\Lambda^{0}_{0} &= \cosh\alpha = \gamma = \frac{1}{\sqrt{1-\beta^2}}.
\end{aligned} \tag{7}$$

For $p^{\mu} = (p^0, \vec{p})$ with $p^0 = E_{\vec{p}}$ and the Lorentz transformation is as follows

$$\begin{aligned}
\vec{p}_{\Lambda} &= [\vec{p} - (\vec{p}\cdot\hat{e})\hat{e}] + [E_{\vec{p}}\sinh\alpha + (\vec{p}\cdot\hat{e})\cosh\alpha]\hat{e},\\
(\Lambda p)^0 &= E_{\vec{p}}\cosh\alpha + (\vec{p}\cdot\hat{e})\sinh\alpha,
\end{aligned} \tag{8}$$

then

$$D^{(1/2)}(\Lambda) = I\cosh\frac{\alpha}{2} + (\vec{\sigma}\cdot\hat{e})\sinh\frac{\alpha}{2}. \tag{9}$$

where I is the unit matrix and $\vec{\sigma}$ is the Pauli matrix. After a lengthy manipulation for a two-component spinor, one can get [7]

$$D^{(1/2)}(W(\Lambda, p)) = \cos\frac{\Omega_{\vec{p}}}{2} + i\sin\frac{\Omega_{\vec{p}}}{2}(\vec{\sigma}\cdot\hat{n}), \tag{10}$$

with

$$\cos\frac{\Omega_{\vec{p}}}{2} = \frac{\cosh\frac{\alpha}{2}\cosh\frac{\beta}{2} + \sinh\frac{\alpha}{2}\sinh\frac{\delta}{2}(\hat{e}\cdot\vec{p})}{[\frac{1}{2} + \frac{1}{2}\cosh\alpha\cosh\delta + \frac{1}{2}\sinh\alpha\sinh\delta(\hat{e}\cdot\vec{p})]^{1/2}} \tag{11}$$

and

$$\sin\frac{\Omega_{\vec{p}}}{2}\hat{n} = \frac{\sinh\frac{\alpha}{2}\sinh\frac{\beta}{2}(\hat{e}\times\vec{p}) + \sinh\frac{\alpha}{2}\sinh\frac{\delta}{2}(\hat{e}\cdot\vec{p})}{[\frac{1}{2} + \frac{1}{2}\cosh\alpha\cosh\delta + \frac{1}{2}\sinh\alpha\sinh\delta(\hat{e}\cdot\vec{p})]^{1/2}}, \tag{12}$$

where $\cosh\delta = (p^0/m)$. Then we can find that (10)–(12) indicate the Lorentz group can be represented by the pure rotation around axis $\hat{n} = \hat{e}\times\vec{p}$ for a two-component spinor.

In this paper we consider the case that the boost is in the x-direction and the momentum vector can be given by

$$\vec{p} = (E_{\vec{p}}, p\cos(\theta), p\sin(\theta)\cos(\varphi), p\sin(\theta)\cos(\varphi)), \tag{13}$$

with $E_{\vec{p}} = \sqrt{p^2 + m^2}$, $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$, the Wigner representation for this boost is given by

$$D(\Lambda_{\vec{p}}) = \begin{pmatrix} \cos(\Omega_{\vec{p}}/2) + i\sin(\Omega_{\vec{p}}/2)\cos(\varphi) & -\sin(\Omega_{\vec{p}}/2)\sin(\varphi) \\ \sin(\Omega_{\vec{p}}/2)\sin(\varphi) & \cos(\Omega_{\vec{p}}/2) - i\sin(\Omega_{\vec{p}}/2)\cos(\varphi) \end{pmatrix} = D(\Omega_{\vec{p}}). \tag{14}$$

Equation (14) gives the explicit representation of a pure Lorentz boost. In the next section using (14) we will give the transformation of an entangled state.

3 Relativistic Entanglement State and Entanglement Concurrence

We investigate a bipartite state in the momentum representation. In the rest frame this state can be defined as follows

$$\Phi(\vec{p},\vec{q}) = \iint d\vec{p}d\vec{q}g(\vec{p},\vec{q})(\sqrt{1-\varepsilon}|\uparrow\uparrow\rangle + \sqrt{\varepsilon}|\downarrow\downarrow\rangle), \tag{15}$$

where $\vec{p}$ and $\vec{q}$ are the momenta for the first and second particles, the momenta distribution $g(\vec{p},\vec{q})$ must satisfy

$$\iint |g(\vec{p},\vec{q})|^2 \tilde{d}\vec{p}\tilde{d}\vec{q} = 1, \tag{16}$$

where $\tilde{d}\vec{p}(\tilde{d}\vec{q})$ is the normalized integration measure given by [6]

$$\tilde{d}\vec{p} = \frac{d^3\vec{p}}{2\sqrt{\vec{p}^2 + m^2}}, \tag{17}$$

and $|\uparrow\uparrow\rangle = |\uparrow\rangle \otimes |\uparrow\rangle$, $|\downarrow\downarrow\rangle = |\downarrow\rangle \otimes |\downarrow\rangle$ with

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In the rest frame, by integrating the momenta we can get the reduced density matrix for spins

$$\rho_{AB} = \iint d\vec{p}d\vec{q}|g(\vec{p},\vec{q})|^2 |\Phi(\vec{p},\vec{q})\rangle_{AB\,BA}\langle\Phi(\vec{p},\vec{q})|. \tag{18}$$

Here we note that in the rest frame there is no entanglement between the spin and the moment parts of this state and the spins are not maximally entangled ($\varepsilon \neq \frac{1}{2}$), while the entanglement between the momenta depends on $g(\vec{p},\vec{q})$. Necessarily we should acknowledge that there is still some ambiguities when we give the explicit representation of $g(\vec{p},\vec{q})$ which can be called "relativistic Gaussian" wavepacket.

Now make a Lorentz transformation. In the view of an observer in the moving frame the entanglement state (15) is given, via (4) and (14), by

$$U(\Lambda)|\Phi(\vec{p},\vec{q})\rangle_{AB} = \iint \tilde{d}\vec{p}\tilde{d}\vec{q}g(\vec{p},\vec{q})\sqrt{\frac{(\Lambda p)^0}{p^0}}\sqrt{\frac{(\Lambda q)^0}{q^0}}|\vec{p}_\Lambda,\vec{q}_\Lambda\rangle \otimes |\Psi(\vec{p},\vec{q})\rangle_{AB}, \tag{19}$$

where A and B denote the two particles and

$$|\Psi(\vec{p},\vec{q})\rangle_{AB} = \sum_{\sigma_A,\sigma_B,\sigma'_A,\sigma'_B} D_{\sigma'_A\sigma_A}(\Omega_{\vec{p}}) \otimes D_{\sigma'_B\sigma_B}(\Omega_{\vec{q}})|\sigma'_A,\sigma'_B\rangle_{A'B'}, \tag{20}$$

and

$$|\sigma'_A,\sigma'_B\rangle_{A'B'} = \sqrt{1-\varepsilon}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \sqrt{\varepsilon}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \tag{21}$$

where σ denotes the spin part of the two particles. For the distribution of the momenta we use an relativistic "entanglement Gaussian" [12] with width w,

$$g(\vec{p},\vec{q}) = \sqrt{\frac{1}{N}\exp\left[-\frac{\vec{p}^2+\vec{q}^2}{4w^2}\right]\exp\left[-\frac{\vec{p}^2+\vec{q}^2-2x\vec{p}\cdot\vec{q}}{4w^2(1-x^2)}\right]}, \tag{22}$$

where $x \in [0, 1)$ and N is the normalization factor. In (22), for a given w , x can be regarded as a measure of the entanglement between the momenta. In the limit $x \to 1$ we have

$$\lim_{x\to 1} g(\vec{p},\vec{q}) = \sqrt{\frac{1}{N'}\exp\left[-\frac{\vec{p}^2}{2w^2}\right]\delta^3(\vec{p}-\vec{q})} = g'(\vec{p},\vec{q}), \tag{23}$$

where N' is the normalization factor. Equation (23) indicates a perfect correlation between the momenta, but the momenta are not necessarily maximally entangled [12]. Note that, here we have used the relativistically invariant δ function

$$\delta(\vec{p}-\vec{q}) = \frac{(\Lambda p)^0}{p^0}\delta(\vec{p}_\Lambda - \vec{q}_\Lambda) \tag{24}$$

and $\sqrt{(\Lambda p)^0/p^0}$ ($\sqrt{(\Lambda q)^0/q^0}$) in (19) can be absorbed into the normalization factor N', so we can rewrite (19) as follows

$$U(\Lambda)|\Phi(\vec{p},\vec{q})\rangle_{AB} = \iint \tilde{d}\vec{p}\tilde{d}\vec{q}\, g'(\vec{p},\vec{q})|\vec{p}_\Lambda,\vec{q}_\Lambda\rangle \otimes |\Psi(\vec{p},\vec{q})\rangle_{AB}, \tag{25}$$

By using (14), (19), (20), (21), (25) and integrating the momenta, we obtain the reduced density matrix for spins, viewed by the Lorentz boost observer as

$$\rho'_{AB} = \iint \tilde{d}\vec{p}\tilde{d}\vec{q}|g'(\vec{p},\vec{q})|^2|\Psi(\vec{p},\vec{q})\rangle_{AB\,BA}\langle\Psi(\vec{p},\vec{q})|. \tag{26}$$

The entanglement between the bipartite state can be obtained by calculating the "concurrence" which was first introduced by Wootters [15]

$$\mathcal{C}(\rho_{AB}) = \max\{\lambda_1-\lambda_2-\lambda_3-\lambda_4, 0\}, \tag{27}$$

where $\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ are the square roots of the eigenvalues of the matrix $\rho_{AB}\tilde{\rho}_{AB}$ and $\tilde{\rho}_{AB}$ is the "time-reversed" matrix which is defined by

$$\tilde{\rho} = (\sigma_y\otimes\sigma_y)\rho^*(\sigma_y\otimes\sigma_y), \tag{28}$$

where ρ^* is the complex conjugation of ρ and $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. After some manipulations we get the concurrence of the state given by (26):

$$\mathcal{C}(\rho'_{AB}) = 2\sqrt{\varepsilon - \varepsilon^2}. \tag{29}$$

In the rest frame we can also get the concurrence of (18)

$$\mathcal{C}(\rho_{AB}) = 2\sqrt{\varepsilon - \varepsilon^2}. \tag{30}$$

which is equal to the result given in (29).

As has been shown in [12], at the limit $x \to 1$ (see (22), (23)) while the momenta are perfectly correlated the concurrence dose not decrease and is independent of w/m and the boost $\Omega_{\vec{p}}(\Omega_{\vec{q}})$. In this paper we argue that if the momentum part of the bipartite state is appropriately entangled there is no transfer of entanglement from spin to momentum or inversely there is no entanglement transfer from momentum to spin. Moreover since the notion "the spin state of a particle" is meaningless if we do not consider the momentum part of the state simultaneously [5]; the reduced density matrix for the spins is invariant only if we consider the momentum eigenstates (plane waves) as well. Because of this, one can develop a relativistic invariant quantum communication protocol [12] that could be applied to the moving conditions.

Here we note that the "entanglement Gaussian" which we use in this paper is only useful in theories [16]. Because the quantum states that one can prepare in real experiments are necessarily localized, the nonlocalized states are not practical in reality. The purity of the spin reduced matrix in the relativistic case depend on how much the spatial wave package is localized, that is to say the more the spatial wave package is localized the more the purity of the reduced spin matrix decreases when viewed from moving frames [6, 12]. The localization of quantum states can be simply regarded as the width of the wave package, so in the limit of a negligible package width the spin-spin entanglement would remain maximal when viewed from moving frames.

4 Conclusion

In this paper we have shown that if the momentum part of a bipartite state is appropriately entangled there is no transfer of entanglement from spin to momentum. Mathematically, Li and Du proved that [16] when a pure state with separable spin and momentum in the rest frame is viewed from a moving reference frame its reduced density matrix for spins necessarily appears to be mixed if its spatial wave package is localized which is in the same line with our result in this paper.

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